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Asymptotic analysis of generalized Hermite polynomials

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Abstract

We analyze the polynomials $H_n^r(x)$ considered by Gould and Hopper, which generalize the classical Hermite polynomials. We present the main properties of $H_n^r(x)$ and derive asymptotic approximations for large values of n from their differential-difference equation, using a discrete ray method. We give numerical examples showing the accuracy of our formulas.

Keywords: Hermite polynomials, asymptotic analysis, ray method, differential-difference equations, discrete WKB method.

MSC-class: 33C45 (Primary) 34E05, 34E20 (Secondary)

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1 Introduction

In [11] Gould and Hopper generalized the classical Hermite polynomials by introducing the polynomials $H_n^r(x,a,p)$ defined by

$$H_n^r(x,a,p) = (-1)^n x^{-a} \exp(px^r) \frac{d^n}{dx^n} \left[x^a \exp(-px^r)\right].$$

They derived several properties of $H_n^r(x,a,p)$, including a generating function, differentiation, addition and operational formulas. As they remarked, the case $a=0$ was considered before them by Bell in [1], where he analyzed the polynomials

$$\xi_n(x,t;r) = \exp(-xt^r) \frac{d^n}{dt^n} \exp(xt^r).$$

Dhawan [5], obtained a new generating function, summation formulas, a hypergeometric representation and some integrals of $H_n^r(x,a,p)$, including

$$\int_0^\infty x^a \exp(-px^r) H_n^r(x,a,p) H_m^s(x,a,p) dx = 0$$

for $n > ms$.

Kalinowski and Seweryński [14], [21], constructed a differential equation of order r for $H_n^r(x,0,p)$. They also proved that the polynomials $H_n^r(x,0,p)$ are not orthogonal with respect to the weight function $\exp(-px^r)$, r even, in the interval $(-\infty,\infty)$.

Todorov [25], considered the polynomials $H_n^r(x,0,p)$ in connection with his analysis of the n^{th} derivative of the composite function $f(x^r)$. Unaware of Kalinowski and Seweryński's previous work, he re-derived the differential equation for $H_n^r(x,0,p)$.

Additional properties and further generalizations where studied by Chatterjea [2], Chongdar [3], Joshi and Prajapat [13], Munot, and Mathur [17], Rajagopal [18], Riordan [19], Saha [20], Shrivastava [22] and Singh and Tiwari [23].

In this work, we will consider the polynomials

$$H_n^r(x) = (-1)^n \exp(x^r) \frac{d^n}{dx^n} \exp(-x^r), \tag{1}$$

with $r = 2, 3, \ldots, \quad n = 0, 1, \ldots,$ which correspond to the particular case $H_n^r(x, 0, 1)$. Clearly $H_n^2(x) = H_n(x) =$ Hermite polynomial of degree n. The first few $H_n^r(x)$ are

$$H_0^r(x) = 1, \quad H_1^r(x) = rx^{r-1}, \quad H_2^r(x) = r^2 x^{2(r-1)} - r\,(r-1)\,x^{r-2}$$
$$H_3^r(x) = r^3 x^{3(r-1)} - 3r^2\,(r-1)\,x^{2r-3} + r\,(r-1)\,(r-2)\,x^{r-3}$$

and in general

$$H_n^r(x) = r^n x^{n(r-1)} - \cdots - (-r)_n \;\; x^{r-n},$$

where $(\cdot)_n$ denotes the Pochhammer symbol. Our work was motivated by the talk "Asymptotics for Hermite type polynomials", delivered by Professor Wolfgang Gawronski at the conference organized in honor of the 65th birthday of Nico M. Temme in Santander, Spain on July 4-6, 2005. In his talk, he considered the behavior of $H_n^r(x)$ and its zeros using Plancherel-Rotach type asymptotics.

In Section 2, we present the basic properties of $H_n^r(x)$. Although some are not new, we present proofs of all of them for completion purposes. In Section 3, we present the asymptotic analysis using a modified ray method, developed by Dosdale, Duggan and Morgan in [8] and formalized by Costin and Costin in [4]. In a previous work [7], we successfully applied the same technique to the classical Hermite polynomials $H_n^2(x)$. Section 4 contains our main result and supporting numerical examples.

2 Properties

The Hermite polynomials admit the simple hypergeometric representation [15]

$$H_n(x) = (2x)^n \;\; {}_2F_0\left(\begin{array}{c} -\frac{n}{2}, -\frac{(n-1)}{2} \\ - \end{array} \middle| -x^{-2} \right),$$

where ${}_pF_q\left[\cdot\right]$ is the hypergeometric function [10]. On the other hand, for $H_n^r(x)$ we need to consider the extension of ${}_pF_q$ given by Meijer's G-function [9].

Proposition 1 *The polynomials $H_n^r(x)$ can be represented in terms of Meijer's G-function by*

$$H_n^r(x) = \exp\left(x^r\right) \left(-\frac{r}{x}\right)^n G_{r,r+1}^{1,r}\left(x^r \middle| \begin{array}{c} 1-\frac{1}{r}, 1-\frac{2}{r}, \cdots, 0 \\ 0, 1+\frac{n-1}{r}, 1+\frac{n-2}{r}, \cdots, \frac{n}{r} \end{array} \right).$$

Proof. We have

$$\frac{d^n}{dx^n}\exp\left(-x^r\right)=\sum_{j=0}^{\infty}\frac{d^n}{dx^n}\frac{\left(-x^r\right)^j}{j!}=\frac{1}{x^n}\sum_{j=0}^{\infty}\frac{(rj)!}{(rj-n)!}\frac{\left(-x^r\right)^j}{j!}. \tag{2}$$

Using the multiplication formula for the Gamma function [24], we obtain

$$\frac{(rj)!}{(rj-n)!}=\frac{\Gamma\left(rj+1\right)}{\Gamma\left(rj-n+1\right)}=\frac{r^{rj+1}\prod_{i=0}^{r-1}\Gamma\left(\frac{i+1}{r}+j\right)}{r^{rj-n+1}\prod_{i=0}^{r-1}\Gamma\left(\frac{i+1-n}{r}+j\right)}$$

$$=r^n\prod_{i=1}^{r}\frac{\Gamma\left(\frac{i}{r}+j\right)}{\Gamma\left(\frac{i-n}{r}+j\right)}=r^n\prod_{i=1}^{r}\frac{\Gamma\left(\frac{i}{r}\right)\left(\frac{i}{r}\right)_j}{\Gamma\left(\frac{i-n}{r}\right)\left(\frac{i-n}{r}\right)_j}. \tag{3}$$

Replacing (3) in (2) we get

$$\frac{d^n}{dx^n}\exp\left(-x^r\right)=\left(\frac{r}{x}\right)^n\prod_{i=1}^{r}\frac{\Gamma\left(\frac{i}{r}\right)}{\Gamma\left(\frac{i-n}{r}\right)}\sum_{j=0}^{\infty}\prod_{i=1}^{r}\frac{\left(\frac{i}{r}\right)_j}{\left(\frac{i-n}{r}\right)_j}\frac{\left(-x^r\right)^j}{j!}$$

$$=\left(\frac{r}{x}\right)^n\prod_{i=1}^{r}\frac{\Gamma\left(\frac{i}{r}\right)}{\Gamma\left(\frac{i-n}{r}\right)}\ {}_rF_r\left[\left.\begin{matrix}\frac{1}{r},\frac{2}{r},\cdots,1\\ \frac{1-n}{r},\frac{2-n}{r},\cdots,\frac{r-n}{r}\end{matrix}\right|-x^r\right].$$

Thus, from the definition of $H_n^r(x)$, it follows that

$$H_n^r(x)=\exp\left(x^r\right)\left(-\frac{r}{x}\right)^n\prod_{i=1}^{r}\frac{\Gamma\left(\frac{i}{r}\right)}{\Gamma\left(\frac{i-n}{r}\right)}\ {}_rF_r\left[\left.\begin{matrix}\frac{1}{r},\frac{2}{r},\cdots,1\\ \frac{1-n}{r},\frac{2-n}{r},\cdots,\frac{r-n}{r}\end{matrix}\right|-x^r\right]. \tag{4}$$

Using the relation between the hypergeometric function and Meijer's G-function [12]

$$\frac{\prod_{i=1}^{p}\Gamma\left(a_i\right)}{\prod_{i=1}^{q}\Gamma\left(b_i\right)}\ {}_pF_q\left[\left.\begin{matrix}a_1,\cdots a_p\\ b_1,\cdots b_q\end{matrix}\right|x\right]=G_{p,q+1}^{1,p}\left(-x\left|\begin{matrix}1-a_1,\cdots 1-a_p\\ 0,1-b_1,\cdots 1-b_q\end{matrix}\right.\right)$$

in (4), the results follow. ■

Proposition 2 *The polynomials $H_n^r(x)$ satisfy the differential-difference equation*

$$H_{n+1}^r(x) + \frac{d}{dx} H_n^r(x) = r x^{r-1} H_n^r(x). \tag{5}$$

Proof. The result follows immediately from the Rodrigues formula (1), since

$$\begin{aligned} \frac{d}{dx} H_n^r(x) &= (-1)^n \, r x^{r-1} \exp(x^r) \frac{d^n}{dx^n} \exp(-x^r) \\ + (-1)^n \exp(x^r) \frac{d^{n+1}}{dx^{n+1}} \exp(-x^r) &= r x^{r-1} H_n^r(x) - H_{n+1}^r(x). \end{aligned}$$

■

When $r = 2$, we recover the well-known formula for the Hermite polynomials [16]

$$H_{n+1}(x) + H_n'(x) = 2x H_n(x).$$

Proposition 3 *The polynomials $H_n^r(x)$ have the exponential generating function*

$$G(x,t) = \sum_{n=0}^{\infty} H_n^r(x) \frac{t^n}{n!} = \exp\left[x^r - (x-t)^r\right]. \tag{6}$$

Proof. From (1) we get

$$\begin{aligned} G(x,t) &= \exp(x^r) \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \left[\frac{d^n}{du^n} e^{-u^r}\right]_{u=x} \\ &= \exp(x^r) \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\frac{d^n}{dt^n} e^{-(x-t)^r}\right]_{t=0} \\ &= \exp(x^r) \exp\left[-(x-t)^r\right]. \end{aligned}$$

■

In particular, for $r = 2$, we have

$$\exp\left[x^2 - (x-t)^2\right] = \exp\left(2tx - t^2\right),$$

which is the exponential generating function of the Hermite polynomials [16].

Proposition 4 *The polynomials $H_n^r(x)$ admit the explicit representation*

$$H_n^r(x) = \sum_{k=\lfloor \frac{n}{r} \rfloor}^{n} C_k^n(r) x^{rk-n}, \tag{7}$$

where

$$C_k^n(r) = \frac{(-1)^n\, n!}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{rj}{n}. \tag{8}$$

Proof. From (1) we have

$$\begin{aligned} H_n^r(x) &= (-1)^n \exp(x^r) \frac{d^n}{dx^n} \exp(-x^r) \\ &= (-1)^n \left[\sum_{k=0}^{\infty} \frac{(x^r)^k}{k!} \right] \left[\sum_{j=0}^{\infty} \frac{d^n}{dx^n} \frac{(-x^r)^j}{j!} \right] \\ &= \frac{(-1)^n\, n!}{x^n} \left[\sum_{k=0}^{\infty} \frac{(x^r)^k}{k!} \right] \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \binom{rj}{n} (x^r)^j \right], \end{aligned}$$

and hence [26, (2.8)]

$$H_n^r(x) = \frac{(-1)^n\, n!}{x^n} \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{rj}{n} \frac{1}{(k-j)!} \right] (x^r)^k .$$

If $rk < n$, we have

$$\binom{rj}{n} = 0, \quad 0 \leq j \leq k$$

and therefore,

$$H_n^r(x) = \frac{(-1)^n\, n!}{x^n} \sum_{k=\lfloor \frac{n}{r} \rfloor}^{\infty} \left[\sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{rj}{n} \frac{1}{(k-j)!} \right] (x^r)^k ,$$

from which (7) follows. ∎

Note that when $r=2$, (7) reduces to the well-known representation of the Hermite polynomials [16]

$$H_n(x)=\sum_{k=\lfloor\frac{n}{2}\rfloor}^{n}\frac{(-1)^n\,n!}{k!}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{2j}{n}\right]x^{2k-n}$$

$$=\sum_{k=\lfloor\frac{n}{2}\rfloor}^{n}(-1)^n\,n!\frac{(-1)^k\,2^{2k-n}}{(2k-n)!(n-k)!}x^{2k-n}=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k\,n!}{(n-2k)!k!}(2x)^{n-2k}.$$

Corollary 5 *The polynomials $H_n^r(x)$ satisfy*

$$H_n^r(x)=\omega^n H_n^r(\omega x),\quad \text{with}\ \ \omega^r=1. \tag{9}$$

The symmetry relation (9) generalizes the reflection formula of the Hermite polynomials [16]

$$H_n(x)=(-1)^n\,H_n(-x).$$

Remark 6 *It follows from (9) that it is enough to analyze $H_n^r(x)$ in the region $|\arg(x)|\leq\frac{\pi}{r}$ of the complex plane. In particular, the roots of $H_n^r(x)$ are completely determined by its positive real roots. To show this, in Figure 1 we plot the roots of $H_5^5(x)$ in the complex plane.*

Proposition 7 *The polynomials $H_n^r(x)$ can be represented by*

$$H_n^r(x)=(-1)^n\,n!\sum_{\substack{N_1+\cdots+N_r=n\\ n\geq 0}}\prod_{j=1}^{r}\Omega_j\left(N_j\right)\binom{r}{j}^{\frac{N_j}{j}}\frac{(-1)^{\frac{N_j}{j}}}{\left(\frac{N_j}{j}\right)!}x^{\frac{(r-j)}{j}N_j}.$$

In particular, we have

$$H_n^r(0)=(-1)^{\frac{r-1}{r}n}\frac{n!}{\left(\frac{n}{r}\right)!}\Omega_r\left(n\right), \tag{10}$$

where [26, (2.32)]

$$\Omega_r\left(k\right)=\frac{1}{r}\sum_{j=1}^{r}\exp\left(\frac{2\pi jk\mathrm{i}}{r}\right)=\begin{cases}1 & \text{if}\quad r|_k\\ 0 & \text{otherwise}\end{cases}. \tag{11}$$

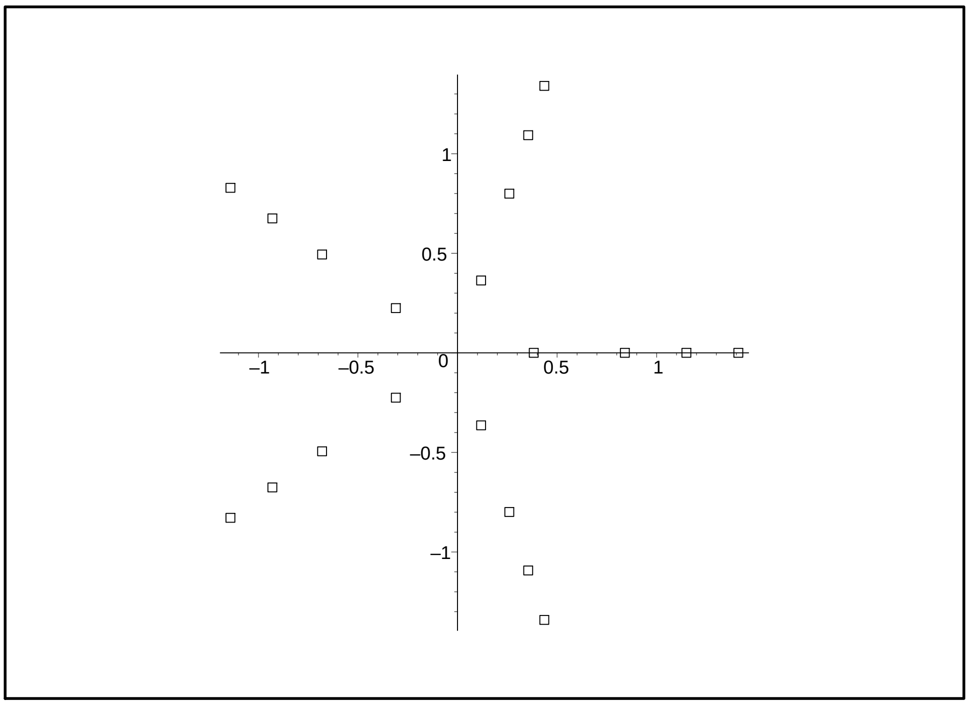


Figure 1: A plot of the zeros of $H_5^5(x)$ in the complex plane.

Proof. From (6) we get

$$\sum_{n=0}^{\infty} \frac{H_n^r(x)}{n!} t^n \tag{12}$$

$$= \exp\left[x^r - (x-t)^r\right] = \exp\left[x^r - \sum_{j=0}^{r} \binom{r}{j} (-1)^j x^{r-j} t^j\right]$$

$$= \exp\left[\sum_{j=1}^{r} \binom{r}{j} (-1)^{j+1} x^{r-j} t^j\right] = \prod_{j=1}^{r} \exp\left[\binom{r}{j} (-1)^{j+1} x^{r-j} t^j\right]$$

$$= \prod_{j=1}^{r} \sum_{N_j=0}^{\infty} \binom{r}{j}^{N_j} \frac{(-1)^{(j+1)N_j}}{(N_j)!} x^{(r-j)N_j} t^{jN_j}$$

$$= \prod_{j=1}^{r} \sum_{N_j=0}^{\infty} \Omega_j(N_j) \binom{r}{j}^{\frac{N_j}{j}} \frac{(-1)^{\frac{(j-1)}{j}N_j}}{\left(\frac{N_j}{j}\right)!} x^{\frac{(r-j)}{j}N_j} t^{N_j}$$

$$= \sum_{\substack{N_1+\cdots+N_r=n \\ n\geq 0}} \left[\prod_{j=1}^{r} \Omega_j(N_j) \binom{r}{j}^{\frac{N_j}{j}} \frac{(-1)^{\frac{(j-1)}{j}N_j}}{\left(\frac{N_j}{j}\right)!} x^{\frac{(r-j)}{j}N_j}\right] t^n, \tag{13}$$

where we have used equation (2.10) in [26].

Comparing (12) and (13), we conclude that

$$H_n^r(x) = n! \sum_{\substack{N_1+\cdots+N_r=n \\ n\geq 0}} \prod_{j=1}^{r} \Omega_j(N_j) \binom{r}{j}^{\frac{N_j}{j}} \frac{(-1)^{\frac{(j+1)}{j}N_j}}{\left(\frac{N_j}{j}\right)!} x^{\frac{(r-j)}{j}N_j}$$

$$= (-1)^n n! \sum_{\substack{N_1+\cdots+N_r=n \\ n\geq 0}} \prod_{j=1}^{r} \Omega_j(N_j) \binom{r}{j}^{\frac{N_j}{j}} \frac{(-1)^{\frac{N_j}{j}}}{\left(\frac{N_j}{j}\right)!} x^{\frac{(r-j)}{j}N_j}$$

In particular, when $x = 0$, we must have

$$N_1 = \cdots = N_{r-1} = 0 \quad \text{and} \quad N_r = n,$$

from which (10) follows. ■

Proposition 8 *The polynomials $H_n^r(x)$ satisfy the recurrence relation*

$$H_{n+1}^r(x) = r \sum_{k=0}^{r-1} (-1)^k \, k! \binom{n}{k} \binom{r-1}{k} x^{r-1-k} H_{n-k}^r(x). \tag{14}$$

Proof. From (6), we have

$$\frac{\partial G}{\partial t} = r \left(x - t\right)^{r-1} G$$

or [Wilf],

$$\sum_{n=0}^{\infty} H_{n+1}^r(x) \frac{t^n}{n!} = r \left[\sum_{k=0}^{\infty} (-1)^k \binom{r-1}{k} x^{r-1-k} t^k \right] \left[\sum_{n=0}^{\infty} H_n^r(x) \frac{t^n}{n!} \right].$$

Comparing coefficients of t we get

$$\frac{H_{n+1}^r(x)}{n!} = r \sum_{k=0}^{n} (-1)^k \binom{r-1}{k} x^{r-1-k} \frac{H_{n-k}^r(x)}{(n-k)!}$$

and (14) follows. ■

When $r = 2$, (14) reduces to the three-term recurrence relation for the Hermite polynomials [16]

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x).$$

Proposition 9 *Let $u_n(x) = \exp\left(-x^r\right) H_n^r(x)$. Then,*

$$u_n^{(r)} + r \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{(n+r-1)!}{(n+k)!} x^k u_n^{(k)} = 0, \tag{15}$$

where

$$u_n^{(k)} = \frac{d^k}{dx^k} u_n\left(x\right).$$

Proof. It is clear from (1) that

$$u_{n+k}(x) = \left(-1\right)^k u_n^{(k)}. \tag{16}$$

From (14), we get

$$
\begin{aligned}
u_{n+r}(x) &= r\sum_{k=0}^{r-1} (-1)^k\, k! \binom{n+r-1}{k} \binom{r-1}{k} x^{r-1-k} u_{n+r-1-k}(x) \\
&= r\sum_{k=0}^{r-1} (-1)^{r-1-k} (r-1-k)! \binom{n+r-1}{r-1-k} \binom{r-1}{r-1-k} x^k u_{n+k}(x).
\end{aligned}
$$

Thus,

$$
u_{n+r}(x) = r\sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1}{k} \frac{(n+r-1)!}{(n+k)!} x^k u_{n+k}(x). \tag{17}
$$

Using (16) in (17), the result follows. ■

For the case $r=2$, (15) gives

$$
\left[\exp\left(-x^2\right) H_n\right]'' + 2x\left[\exp\left(-x^2\right) H_n\right]' + 2(n+1)\exp\left(-x^2\right) H_n = 0,
$$

which is equivalent to the differential equation of the Hermite polynomials [16]

$$
H_n'' - 2xH_n' + 2nH_n = 0.
$$

3 Asymptotic analysis

We seek an approximative solution for (5) of the form

$$
H_n^r(x) \sim \exp\left[f(x,n) + g(x,n)\right], \quad n \to \infty \tag{18}
$$

with

$$
g = o(f), \quad n \to \infty.
$$

Since $H_0^r(x) = 1$, we must have

$$
f(x,0) = 0 \quad \text{and} \quad g(x,0) = 0. \tag{19}
$$

Using (18) in (5), we have

$$
\begin{aligned}
&\exp\left(f + \frac{\partial f}{\partial n} + \frac{1}{2}\frac{\partial^2 f}{\partial n^2} + g + \frac{\partial g}{\partial n}\right) \qquad (20) \\
&+ \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\exp\left(f+g\right) = rx^{r-1}\exp\left(f+g\right),
\end{aligned}
$$

where we have used

$$f(x,n+1) = f(x,n) + \frac{\partial f}{\partial n}(x,n) + \frac{1}{2}\frac{\partial^2 f}{\partial n^2}(x,n) + \cdots .$$

From (20) we obtain, to leading order, the *eikonal* equation

$$\exp\left(\frac{\partial f}{\partial n}\right) + \frac{\partial f}{\partial x} - r x^{r-1} = 0, \tag{21}$$

and

$$\exp\left(\frac{1}{2}\frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n}\right) + \frac{\partial g}{\partial x}\exp\left(-\frac{\partial f}{\partial n}\right) - 1 = 0,$$

or, to leading order, the *transport* equation

$$\frac{1}{2}\frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x}\exp\left(-\frac{\partial f}{\partial n}\right) = 0. \tag{22}$$

3.1 The rays

To solve (21), we use the method of characteristics, which we briefly review. Given the first order partial differential equation

$$F\left(x,n,f,p,q\right) = 0, \quad \text{with} \quad p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial n},$$

we search for a solution $f(x,n)$ by solving the system of "characteristic equations"

$$\begin{aligned}
&\frac{dx}{dt} = \frac{\partial F}{\partial p}, \quad \frac{dn}{dt} = \frac{\partial F}{\partial q},\\
&\frac{dp}{dt} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial f}, \quad \frac{dq}{dt} = -\frac{\partial F}{\partial n} - q\frac{\partial F}{\partial f},\\
&\frac{df}{dt} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q},
\end{aligned}$$

with initial conditions

$$F\left[x(0,s), n(0,s), f(0,s), p(0,s), q(0,s)\right] = 0, \tag{23}$$

and

$$\frac{d}{ds}f(0,s) = p(0,s)\frac{d}{ds}x(0,s) + q(0,s)\frac{d}{ds}n(0,s), \tag{24}$$

where we now consider $\{x, n, f, p, q\}$ to all be functions of the variables t and s.

For the eikonal equation (21), we have

$$F(x, n, f, p, q) = e^q + p - rx^{r-1} \tag{25}$$

and therefore the characteristic equations are

$$\frac{dx}{dt} = 1, \quad \frac{dn}{dt} = e^q, \quad \frac{dp}{dt} = r(r-1)x^{r-2}, \quad \frac{dq}{dt} = 0, \tag{26}$$

and

$$\frac{df}{dt} = p + qe^q. \tag{27}$$

Solving (26) subject to the initial conditions

$$x(0, s) = s, \quad n(0, s) = 0, \quad q(0, s) = A(s),$$

with $A(s)$ to be determined, we obtain

$$x = t + s, \quad n = te^A, \quad p = r(t+s)^{r-1} + B(s), \quad q = A,$$

for some function $B(s)$. From (23) we have

$$e^A + rs^{r-1} + B - rs^{r-1} = 0$$

and $B = -e^A$. Thus,

$$x = t + s, \quad n = te^A, \quad p = r(t+s)^{r-1} - e^A, \quad q = A. \tag{28}$$

Since (19) implies that $f(0, s) = 0$, we have from (24) and (28)

$$\left(rs^{r-1} - e^A\right) \times 1 + A \times 0 = 0.$$

Hence, $A = \ln\left(rs^{r-1}\right)$ and therefore

$$x = t + s, \quad n = rs^{r-1}t, \tag{29}$$

$$p = r\left[(t+s)^{r-1} - s^{r-1}\right], \quad q = \ln\left(rs^{r-1}\right), \tag{30}$$

with $s > 0$.

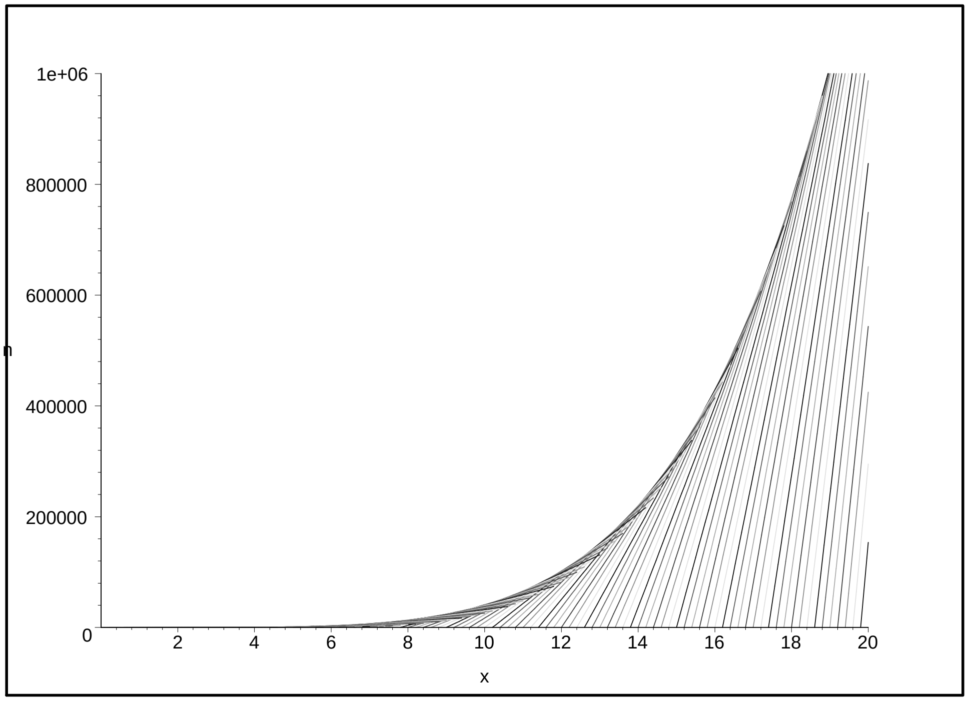


Figure 2: A sketch of the rays with $r = 5$ showing the caustic.

3.2 The caustic

Sketching the rays (29), we observe that they fill the region $x > X_c(n)$, where $X_c(n)$ is the *caustic,* i.e., the points in the (x, n)-plane at which the Jacobian

$$J(t, s) = \frac{\partial x}{\partial t}\frac{\partial n}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial n}{\partial t} = rs^{r-2}\left[(r-1)t - s\right] \tag{31}$$

is zero (see Fig 2).

From (31) we have, for $s > 0$,

$$J(t, s) = 0 \Leftrightarrow s = (r-1)t. \tag{32}$$

Using (32) in (29) we obtain

$$J(t, s) = 0 \Leftrightarrow t = \frac{x}{r} \Leftrightarrow s = \lambda x \tag{33}$$

with

$$\lambda = \frac{r-1}{r}, \quad \frac{1}{2} \le \lambda < 1. \tag{34}$$

From (32) and (33) we conclude that

$$X_c\,(n) = \lambda^{-\lambda} n^{1-\lambda}. \tag{35}$$

3.3 The functions f and g

Using (30) in (27) we have

$$\frac{df}{dt} = r\left[(t+s)^{r-1} - s^{r-1}\right] + \ln\left(rs^{r-1}\right) rs^{r-1}, \tag{36}$$

while (19) implies that $f(0,s) = 0$. Solving (36), we obtain

$$f(t,s) = (t+s)^r - s^r + \left[\ln\left(rs^{r-1}\right) - 1\right] rs^{r-1}t \tag{37}$$

or, using (29),

$$f = x^r - (x-t)^r + n\left[\ln\left(\frac{n}{t}\right) - 1\right]. \tag{38}$$

To solve the transport equation (22), we need to compute $\frac{\partial^2 f}{\partial n^2}$, $\frac{\partial g}{\partial n}$ and $\frac{\partial g}{\partial x}$ as functions of t and s. Use of the chain rule gives

$$\begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial n}{\partial t} & \frac{\partial n}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial n} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and hence,

$$\begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial n} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial n} \end{bmatrix} = \frac{1}{J(t,s)} \begin{bmatrix} \frac{\partial n}{\partial s} & -\frac{\partial x}{\partial s} \\ -\frac{\partial n}{\partial t} & \frac{\partial x}{\partial t} \end{bmatrix}, \tag{39}$$

where the Jacobian $J(t,s)$ was defined in (31). Using (29) and (31) in (39) we have

$$\begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial n} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial n} \end{bmatrix} = \frac{1}{(r-1)t - s} \begin{bmatrix} (r-1)\,t & -\frac{s^{2-r}}{r} \\ -s & \frac{s^{2-r}}{r} \end{bmatrix}. \tag{40}$$

From (37) and (40) we get

$$\begin{aligned} \frac{\partial^2 f}{\partial n^2} &= \frac{\partial^2 f}{\partial n \partial t}\frac{\partial t}{\partial n} + \frac{\partial^2 f}{\partial n \partial s}\frac{\partial s}{\partial n} = \frac{r-1}{rs^{r-1}\left[(r-1)t - s\right]}, \\ \frac{\partial g}{\partial n} &= \frac{\partial g}{\partial t}\frac{\partial t}{\partial n} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial n} = \frac{\frac{\partial g}{\partial s} - \frac{\partial g}{\partial t}}{rs^{r-2}\left[(r-1)t - s\right]}, \\ \frac{\partial g}{\partial x} &= \frac{\partial g}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial x} = \frac{(r-1)\,t\frac{\partial g}{\partial t} - s\frac{\partial g}{\partial s}}{(r-1)t - s}. \end{aligned} \tag{41}$$

Using (41) in (22), we obtain the ODE

$$\frac{\partial g}{\partial t} = -\frac{r-1}{2\left[(r-1)t - s\right]}, \tag{42}$$

and (19) gives $g(0,s) = 0$. Solving (42) we get

$$g(t,s) = \frac{1}{2}\ln\left[\frac{s}{s-(r-1)t}\right] \tag{43}$$

or, using (29),

$$g = \frac{1}{2}\ln\left(\frac{x-t}{x-rt}\right) \tag{44}$$

Note that g is undefined when $x = rt$, i.e., for $x = X_c(n)$.

Thus, for $x > X_c(n)$ we have

$$H_n^r(x) \sim \exp\left[x^r - (x-t)^r + n\ln\left(\frac{n}{t}\right) - n\right]\sqrt{\frac{x-t}{x-rt}}, \quad n \to \infty \tag{45}$$

with $t(x,n)$ defined implicitly by

$$r\left(x-t\right)^{r-1} t - n = 0. \tag{46}$$

3.4 The function $t(x,n)$

To solve (46), we shall use Lagrange's inversion formula.

Theorem 10 *Let $\psi(u)$ and $\phi(u)$ be formal power series in u, with $\phi(0) = 1$. Then there is a unique formal power series $u = u(z)$ that satisfies*

$$u = z\phi(u). \tag{47}$$

Further, we have

$$\left[z^k\right]\left\{\psi\left[u(z)\right]\right\} = \frac{1}{k}\left[u^{k-1}\right]\left\{\psi'(u)\phi(u)^k\right\}, \tag{48}$$

where by $\left[z^k\right]\left\{\psi(z)\right\}$ we mean the coefficient of z^k in the power series of $\psi(z)$.

Proof. See [26, Theorem 5.1]. ■

We first rearrange (46) so that it looks like (47) and obtain

$$\frac{t}{x} = \frac{n}{rx^r}\left(1 - \frac{t}{x}\right)^{1-r}.$$

Using (48) we then have

$$\begin{aligned}\left[\left(\frac{n}{rx^r}\right)^k\right]\left\{\frac{t}{x}\right\} &= \frac{1}{k}\left[u^{k-1}\right]\left\{\left(1-\frac{t}{x}\right)^{(1-r)k}\right\} \\ &= \frac{1}{k}\left[\left(\frac{t}{x}\right)^{k-1}\right]\sum_{j=0}^{\infty}\binom{(1-r)\,k}{j}(-1)^j\left(\frac{t}{x}\right)^j \\ &= \frac{1}{k}\binom{(1-r)\,k}{k-1}(-1)^{k-1}.\end{aligned}$$

Thus,

$$\frac{t}{x} = \sum_{k=1}^{\infty}\frac{1}{k}\binom{(1-r)\,k}{k-1}(-1)^{k-1}\left(\frac{n}{rx^r}\right)^k$$

or

$$t_{\text{out}}(x,n) = x\sum_{k=1}^{\infty}\frac{1}{k}\binom{(1-r)\,k}{k-1}(-1)^{k-1}\left(\frac{n}{rx^r}\right)^k = \lambda x\left[1-\rho\left(\frac{n}{rx^r}\right)\right], \quad (49)$$

where λ was defined in (34) and

$$\rho(z) = \sum_{k=0}^{\infty}\binom{rk}{k}\frac{1}{1-rk}z^k. \quad (50)$$

To find the region of the complex plane where $\rho(z)$ is analytic, we compute its radius of convergence using the ratio test. We have

$$\lim_{k\to\infty}\frac{\binom{rk}{k}\frac{1}{1-rk}}{\binom{r(k+1)}{k+1}\frac{1}{1-r(k+1)}} = \frac{\lambda^r}{r-1},$$

where we have used Stirling's formula [6] and (34)

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}z^z e^{-z}, \quad z\to\infty.$$

Thus, we conclude that $t_{\text{out}}(x,n)$ is analytic for

$$\left|\frac{n}{rx^r}\right| < \frac{\lambda^r}{r-1}$$

or $|x| > X_c(n)$, with $X_c(n)$ defined in (35). Note that from (49) we have

$$t_{\text{out}}(\omega x, n) = \omega t_{\text{out}}(x,n), \quad \text{for} \quad \omega^r = 1.$$

Proposition 11 *The function $\rho(z)$ satisfies the following properties:*

1. *We can represent $\rho(z)$ as a hypergeometric function*

$$\rho(z) = {}_{r-1}F_{r-2}\left[\begin{array}{c} \frac{-1}{r}, \frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-2}{r} \\ \frac{1}{r-1}, \frac{2}{r-1}, \ldots, \frac{r-2}{r-1} \end{array} \middle| \frac{(r-1)}{\lambda^r} z\right], \quad |z| < \frac{\lambda^r}{r-1}. \quad (51)$$

2. *In particular, for $r = 2$, we have*

$$\rho(z) = \sqrt{1-4z}, \quad |z| < \frac{1}{4}$$

and for $r = 3$,

$$\rho(x,n) = \cos\left[\frac{2}{3}\arcsin\left(\frac{3}{2}\sqrt{3z}\right)\right], \quad |z| < \frac{4}{27}.$$

Proof.

1. We have

$$\begin{aligned}
&{}_{r-1}F_{r-2}\left[\begin{array}{c} \frac{-1}{r}, \frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-2}{r} \\ \frac{1}{r-1}, \frac{2}{r-1}, \ldots, \frac{r-2}{r-1} \end{array} \middle| \frac{(r-1)}{\lambda^r} z\right] \\
&= \sum_{k=0}^{\infty}\prod_{j=0}^{r-3}\frac{\left(\frac{j+1}{r}\right)_k}{\left(\frac{j+1}{r-1}\right)_k}\frac{\left(-\frac{1}{r}\right)_k}{(1)_k}\left[\frac{(r-1)}{\lambda^r}z\right]^k
\end{aligned}$$

$$= \sum_{k=0}^{\infty}\frac{\prod\limits_{j=0}^{r-1}\left(\frac{j+1}{r}\right)_k}{\prod\limits_{j=0}^{r-2}\left(\frac{j+1}{r-1}\right)_k}\frac{\left(-\frac{1}{r}\right)_k}{\left(\frac{r-1}{r}\right)_k(1)_k}\left[\frac{(r-1)}{\lambda^r}z\right]^k$$

$$= \sum_{k=0}^{\infty}\prod_{j=0}^{r-1}\frac{\Gamma\left(\frac{j+1}{r}+k\right)}{\Gamma\left(\frac{j+1}{r}\right)}\prod_{j=0}^{r-2}\frac{\Gamma\left(\frac{j+1}{r-1}\right)}{\Gamma\left(\frac{j+1}{r-1}+k\right)}\frac{1}{(1-rk)\,k!}\left[\frac{(r-1)}{\lambda^r}z\right]^k$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(rk+1)}{\Gamma[(r-1)k+1]} \left[\frac{(r-1)^{r-1}}{r^r}\right]^k \frac{1}{(1-rk)\,k!} \left[\frac{(r-1)}{\lambda^r} z\right]^k$$
$$= \sum_{k=0}^{\infty} \binom{rk}{k} \left(\frac{\lambda^r}{r-1}\right)^k \frac{1}{(1-rk)} \left[\frac{(r-1)}{\lambda^r} z\right]^k = \rho(z).$$

2. If $r = 2$, we have
$$\rho(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{1-2k} z^k. \tag{52}$$

Using the identity [26, (2.43)]
$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = (1-4x)^{-\frac{1}{2}},$$

we have
$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{1-2k} x^{-2k} = \frac{1}{x} \int^x \sum_{k=0}^{\infty} \binom{2k}{k} x^{-2k}$$
$$= \frac{1}{x} \int^x \left(1-4u^{-2}\right)^{-\frac{1}{2}} du = \frac{1}{x}\sqrt{x^2-4},$$

or
$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{1-2k} x^k = \sqrt{1-4x}$$

and (52) follows.

The case $r = 3$ was computed using (51) and Maple 10.

■

We shall now find a representation for $t(x,n)$ in the disc $|x| < X_c(n)$. We first observe that when $x = 0$, we have from (46)
$$r(-1)^{r-1}[t(0,n)]^r - n = 0, \tag{53}$$

which implies that $t(0,n) = \tau_l(n)$, where
$$\tau_l(n) = [n(1-\lambda)]^{1-\lambda} \exp[\lambda(2l+1)\pi \mathrm{i}], \quad 1 \le l \le r. \tag{54}$$

Using (53) in (46), we get

$$\frac{(t-x)^{r-1}\,t}{(\tau_l)^r}=1,$$

or

$$\frac{t}{\tau_l}\left(\frac{t}{\tau_l}-\frac{x}{\tau_l}\right)^{r-1}=1. \tag{55}$$

To solve (55) we use the following Lemma.

Lemma 12 *Given an algebraic equation of the form*

$$a\,(a-b)^{c-1}=1,\quad c\neq 0, \tag{56}$$

we formally have

$$a=\sum_{k=0}^{\infty}\frac{1}{1-k}\binom{\frac{k-1}{c}}{k}b^k.$$

Proof. Solving for b in (56) we get

$$b=a-a^{\frac{1}{1-c}}.$$

Letting $\xi=a^{\frac{c}{c-1}}-1$, we have $a=(\xi+1)^{\frac{c-1}{c}}$ and therefore,

$$b=(\xi+1)^{\frac{c-1}{c}}-(\xi+1)^{-\frac{1}{c}}=\xi\,(\xi+1)^{-\frac{1}{c}}$$

or

$$\xi=b\,(\xi+1)^{\frac{1}{c}}. \tag{57}$$

Applying Theorem 10 to (57) with $\phi(\xi)=(\xi+1)^{\frac{1}{c}}$ and $\psi(\xi)=(\xi+1)^{\frac{c-1}{c}}$, we obtain

$$\begin{aligned}\left[b^k\right]\{a\,(b)\}&=\left[b^k\right]\{\psi\,[\xi\,(b)]\}=\frac{1}{k}\left[\xi^{k-1}\right]\left\{\frac{c-1}{c}\,(\xi+1)^{-\frac{1}{c}}\,(\xi+1)^{\frac{k}{c}}\right\}\\&=\frac{c-1}{c}\frac{1}{k}\left[\xi^{k-1}\right]\left\{(\xi+1)^{\frac{k-1}{c}}\right\}=\frac{c-1}{c}\frac{1}{k}\binom{\frac{k-1}{c}}{k-1}=\frac{1}{1-k}\binom{\frac{k-1}{c}}{k}\end{aligned}$$

and the result follows. ■

Thus, applying the Lemma to (55) we find that

$$\frac{t}{\tau_l} = \sum_{j=0}^{\infty} \frac{1}{1-j} \binom{\frac{j-1}{r}}{j} \left(\frac{x}{\tau_l}\right)^j,$$

which we can write as

$$t_{\text{in}}(x, n; l) = \tau_l(n) + \lambda x \mu \left[\frac{x}{\tau_l(n)}\right], \quad 1 \le l \le r, \tag{58}$$

with

$$\mu(z) = \sum_{k=0}^{\infty} \binom{\frac{k}{r}}{k} \frac{1}{k+1} z^k. \tag{59}$$

Applying the ratio test to (59), we get

$$\lim_{k\to\infty} \frac{\binom{\frac{k}{r}}{k}^{\frac{1}{k+1}}}{\binom{\frac{k+1}{r}}{k+1}^{\frac{1}{k+2}}} = \lambda^{-\lambda} (1-\lambda)^{\lambda-1}.$$

From (54) we have

$$|\tau_l(n)| = [n(1-\lambda)]^{1-\lambda}$$

and therefore $t_{\text{in}}(x, n; l)$ is analytic in the sector

$$|x| < [n(1-\lambda)]^{1-\lambda} \lambda^{-\lambda} (1-\lambda)^{\lambda-1} = n^{1-\lambda}\lambda^{-\lambda} = X_c(n).$$

The function $\mu(z)$ doesn't have a simple expression in terms of elementary functions, except for $r = 2$, when we have

$$\mu(z) = 1 + \frac{x}{2 + \sqrt{4 + x^2}}.$$

4 Summary and numerical results

We summarize the results of the previous section in the following theorem. Although our analysis was done on the positive real axis, we can extend our results to the whole complex plane, with the exception of the caustic circle $|x| = X_c(n) = \lambda^{-\lambda} n^{1-\lambda}$.

Theorem 13 *Let*

$$H_n^r(x) = (-1)^n \exp(x^r) \frac{d^n}{dx^n} \exp(-x^r).$$

Then, as $n \to \infty$,

1. *For* $|x| > X_c(n)$, *we have*

$$H_n^r(x) \sim H_{out}(x,n) = \exp\left(x^r \left\{1 - \left[1 - \lambda + \lambda\rho\left(\frac{n}{rx^r}\right)\right]^r\right\} - n\right)$$
$$\times \left\{\frac{n}{\lambda x \left[1 - \rho\left(\frac{n}{rx^r}\right)\right]}\right\}^n \sqrt{\frac{1 - \lambda\left[1 - \rho\left(\frac{n}{rx^r}\right)\right]}{1 - (r-1)\left[1 - \rho\left(\frac{n}{rx^r}\right)\right]}},$$

with $\lambda = \frac{r-1}{r}$ *and*

$$\rho(z) = \sum_{k=0}^{\infty} \binom{rk}{k} \frac{1}{1 - rk} z^k.$$

2. *For* $|x| < X_c(n)$, *the function* $t(x,n)$ *is multivalued and therefore we need to add all the different contributions,*

$$H_n^r(x) \sim H_{in}(x,n) = \sum_{l=1}^{r} \exp\left(x^r - \left\{x - \tau_l(n) - \lambda x \mu\left[\frac{x}{\tau_l(n)}\right]\right\}^r - n\right)$$
$$\times \left\{\frac{n}{\tau_l(n) + \lambda x \mu\left[\frac{x}{\tau_l(n)}\right]}\right\}^n \sqrt{\frac{\tau_l(n) - x\left\{1 - \lambda\mu\left[\frac{x}{\tau_l(n)}\right]\right\}}{r\tau_l(n) - x\left\{1 - (r-1)\,\mu\left[\frac{x}{\tau_l(n)}\right]\right\}}},$$

with

$$\tau_l(n) = [n(1-\lambda)]^{1-\lambda} \exp[\lambda(2l+1)\pi \mathrm{i}]$$

and

$$\mu(z) = \sum_{k=0}^{\infty} \binom{\frac{k}{r}}{k} \frac{1}{k+1} z^k.$$

In Figure 3 we sketch the ratio $\frac{H_n^r(x)}{H_{\text{out}}(x,n)}$, for $r = 5$, $n = 5$ and $X_c(5) \simeq 1.649 < |x| < 7$. We clearly see how the approximation breaks down inside the caustic region $|x| < X_c(n)$.

In Figure 4(a) we compare the values of $H_n^r(x)$ and $H_{\text{in}}(x,n)$ for $r = 5$, $n = 5$ and $|x| < X_c(5)$. To show in detail the graphs for values of x close to

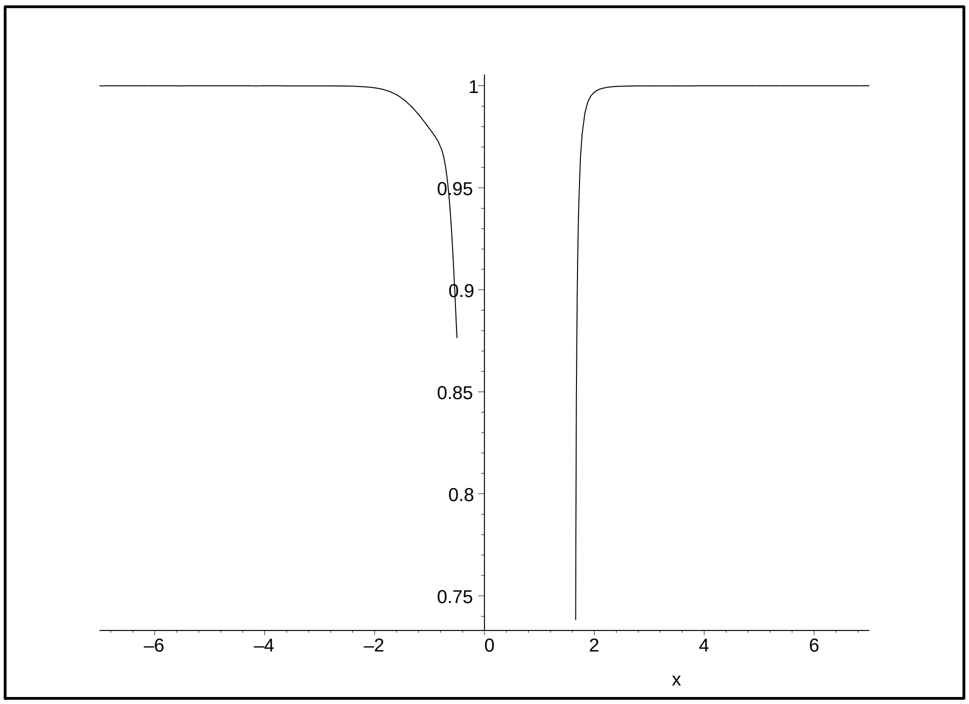


Figure 3: A sketch of $\frac{H_5^5(x)}{H_{\mathrm{out}}(x,5)}$ in the outer region $X_c(5) < |x|$.

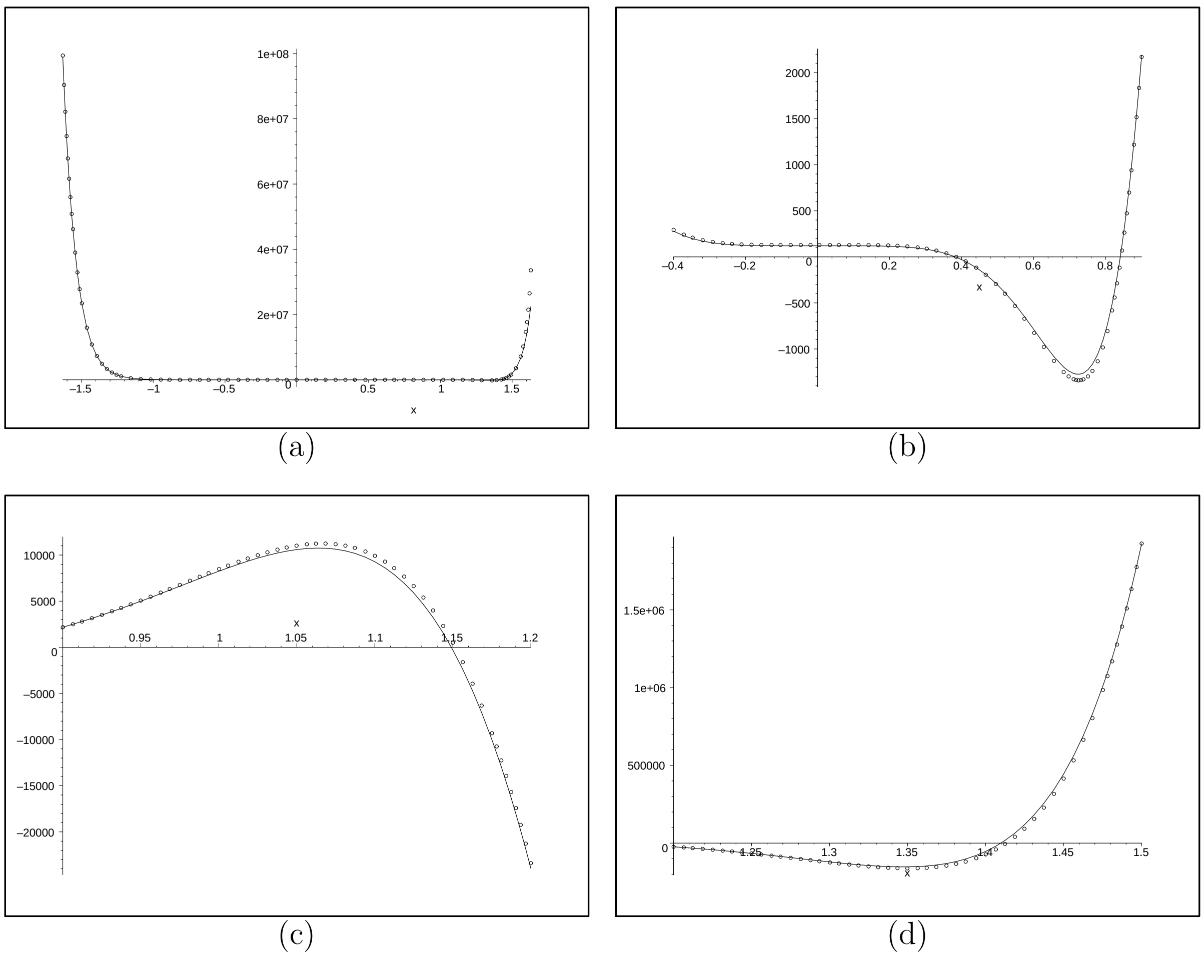
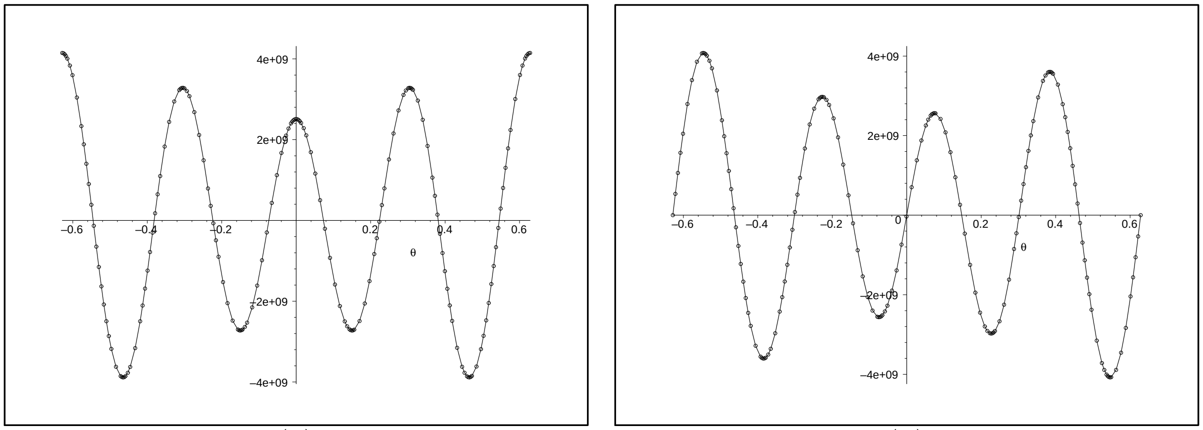


Figure 4: A comparison of $H_5^5(x)$ (solid curve) and $H_{\rm in}(x,5)$ (ooo) inside the caustic region $|x| < X_c(5)$.



(a) (b)

Figure 5: A comparison of $H_5^5(x)$ (solid curve) and $H_{\mathrm{out}}(x, 5)$ (ooo) with $x = 2e^{\mathrm{i}\theta}$.

the zeros of $H_5^5(x)$, we sketch the two functions in smaller intervals in Figures 4(b) , 4(c) and 4(d).

As we mention at the beginning of this section, our approximations are valid in the complex plane minus the circle $|x| = X_c(n)$. To illustrate this, in Figure 5 we graph the real (a) and imaginary (b) parts of $H_n^r(x)$ and $H_{\mathrm{out}}(x, n)$ for $r = 5$, $n = 5$ and $x = 2e^{\mathrm{i}\theta}$. From Corollary 5 we know that it is sufficient to consider the sector $|\theta| < \frac{\pi}{5}$. Finally, in Figure 6 we do the same for the functions $H_n^r(x)$ and $H_{\mathrm{out}}(x, n)$, with $r = 5$, $n = 5$ and $x = \frac{1}{2}e^{\mathrm{i}\theta}$.

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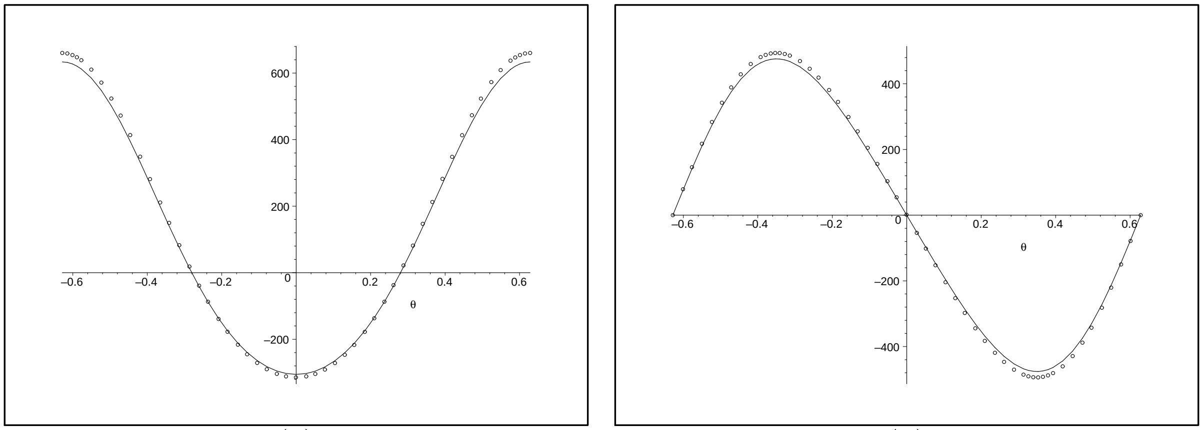

(a) (b)

Figure 6: A comparison of $H_5^5(x)$ (solid curve) and $H_{\text{in}}(x, 5)$ (ooo) with $x = 0.5e^{\mathrm{i}\theta}$.

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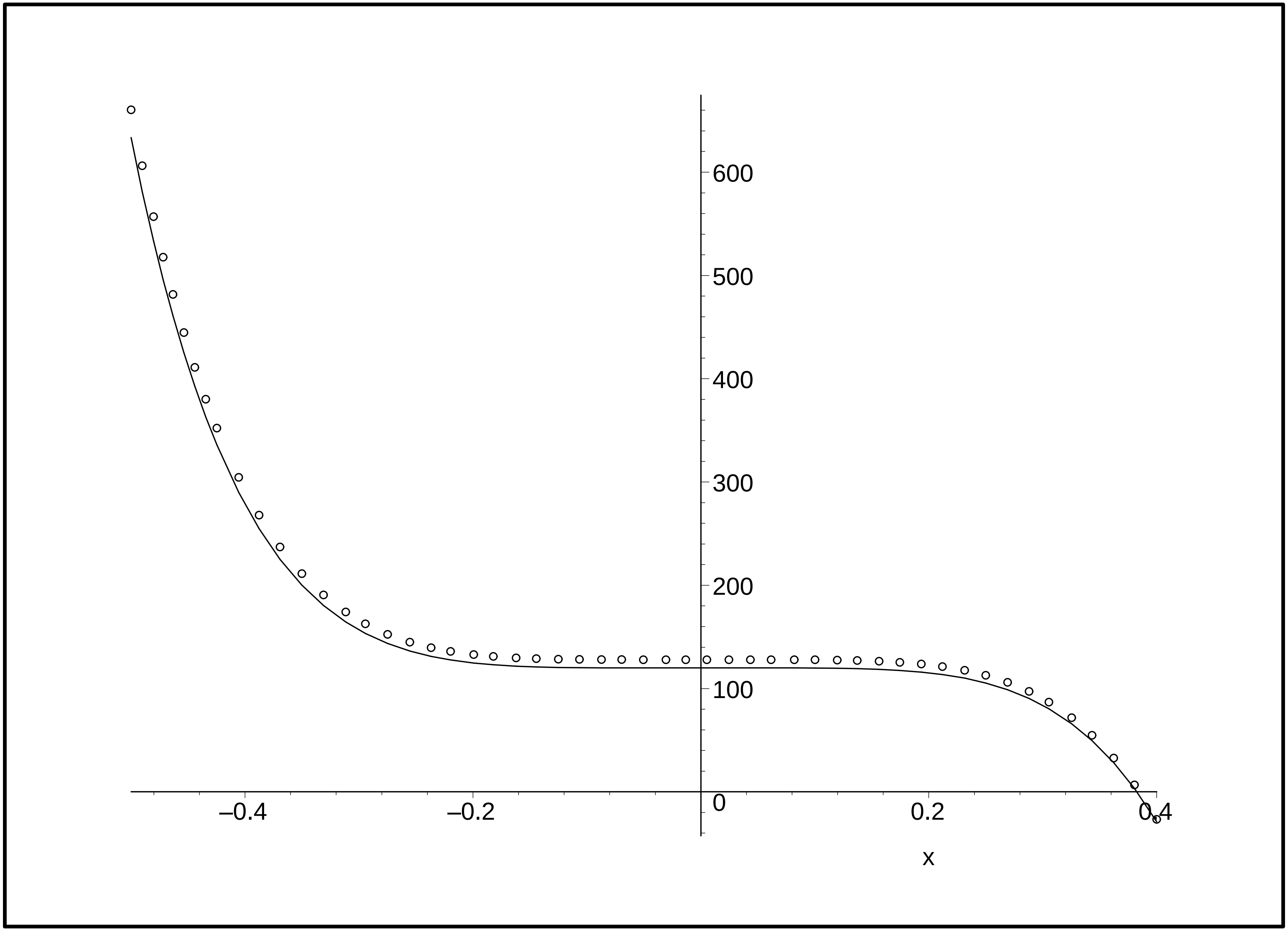
600
500
400
300
200
100
0
–0.4
–0.2
0.2
0.4
x